Rossmoyne Senior High School

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

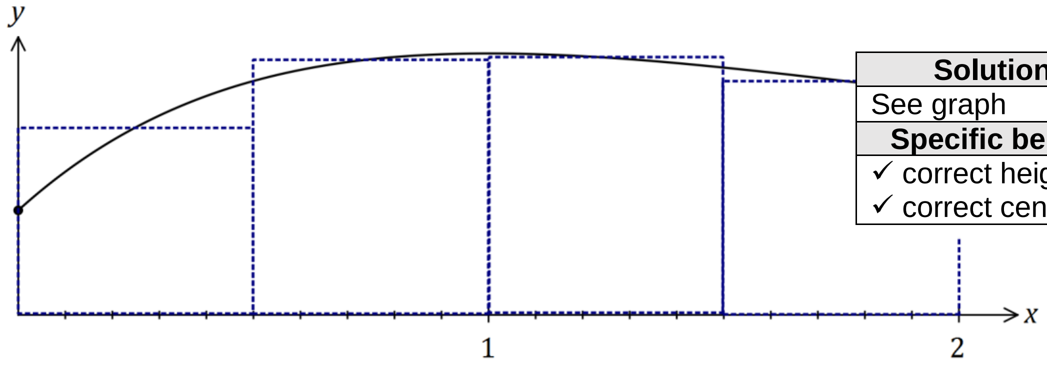
Section Two: Calculator-assumed **65% (97 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

The graph of $y = f(x)$ is shown below for $0 \le x \le 2$, where $f(x) = 2 + 3xe^{1-x}$.



Solution (b)(i)
See graph
Specific behaviours
✓ correct heights
✓ correct centres

(a) Show that $f(x)$ has a stationary point at $(1, 5)$. (3 marks)

Solution
$f'(x) = 3(1-x)e^{1-x}$ $f'(x) = 0$ when $x = 1$ $f(1) = 2 + 3e^0 = 5$ Hence stationary point at $(1, 5)$
Specific behaviours
✓ shows derivative   ✓ shows $f'(x)$ has a factor of $(1-x)$ ✓ indicates $f(1)$

(b) An estimate for the area bounded by the curve, the x-axis, the y-axis and the line $x = 2$ is required. A suitable estimate can be calculated from the sum of the areas of four centred rectangles with heights $f(0.25)$, $f(0.75)$, $f(1.25)$ and $f(1.75)$, each with a width of 0.5 units.

(i) Clearly show these four rectangles on the graph above. (2 marks)

(ii) Use the rectangles to estimate the area, giving your answer correct to 2 decimal places. (3 marks)

Solution
$A = 0.5(3.5878 + 4.8891 + 4.9205 + 4.4799)$ $A = 8.9386 \approx 8.94$ (2dp)
Specific behaviours
✓ indicates correct heights   ✓ multiplies by width   ✓ correct area

Question 10 **(8 marks)**

The capacity, X mL, of glass bottles made in a factory can be modelled by a normal distribution with mean μ and standard deviation 6.5 mL.

(a) If $\mu = 665$, determine

(i) $P(X \leq 660)$. (1 mark)

Solution
$P = 0.2209$
Specific behaviours
✓ states probability

(ii) $P(X > 650 \mid X < 660)$. (2 marks)

Solution
$P(650 < X < 660) = 0.2104$ $P = \frac{0.2104}{0.2209} = 0.9524$
Specific behaviours
✓ calculates numerator ✓ states probability

(iii) the value of x, if $P(X > x) = \frac{1}{7}$. (1 mark)

Solution
$x = 671.9$
Specific behaviours
✓ states value

(b) Given that $P(X \leq k) = 0.983$,

(i) determine the value of μ in terms of k. (3 marks)

Solution
$\frac{k - \mu}{6.5} = 2.12$ $\mu = k - 13.78$
Specific behaviours
✓ Calculates the z-score ✓ Uses the equation for z-scores ✓ expression for μ, correct to 1 dp

(ii) determine μ if $k = 775$. (1 mark)

Solution
$\mu = 761.22$ mL
Specific behaviours
✓ states value

Question 11 **(9 marks)**

Birthday crackers are meant to contain a printed joke. However it is found that in a box of 100, 2% are blank.

(a) Identify the probability distribution of X = the number of blank jokes in a box of crackers and also give the mean and standard deviation. (3 marks)

Solution
$X \sim B(100, 0.02)$ $E(X) = 100 \times 0.02 = 2$ $\sigma_x = \sqrt{2(1-0.02)} = 1.4$
Specific behaviours
✓ indicates binomial distribution with parameters ✓ calculates the expected value ✓ calculates the standard deviation

(b) Determine the probability that there are at least 5 blanks in a randomly selected box. (2 marks)

Solution
$P(X \geq 5) = 0.0508$
Specific behaviours
✓ writes P(X ≥ 5) ✓ calculates the probability

Samples of 20 boxes are collected and the number of blanks recorded.

(c) Determine a 90% confidence interval for the proportion of blanks in a sample of 20 boxes, assuming that 2% are blank. (2 marks)

Solution
$n = 2000 \qquad p = 0.02$ $0.02 \pm 1.645\sqrt{\frac{0.02(1-0.02)}{2000}}$ $[0.0149, 0.0251]$
Specific behaviours
✓ uses k = 1.645 ✓ calculates the correct upper and lower boundaries

(d) Using your 90% confidence interval from part (c), determine the range in which the expected number of blanks in a sample of 20 boxes would lie. (2 marks)

Solution
$0.0149 \times 2000 = 29.8$ $0.0251 \times 2000 = 50.2$ The range of the expected number of blanks is $[30, 50]$
Specific behaviours
✓ Calculates the lowest expected number of blanks ✓ Calculates the highest expected number of blanks

Question 12 **(9 marks)**

A fair die has one face numbered 3, two faces numbered 2 and three faces numbered 1.

(a) Determine the probability that the second odd number occurs on the fourth throw of the die. (3 marks)

Solution
$P(\text{1 odd in 3 throws}) = \frac{2}{3} \times \left(\frac{1}{3}\right)^2 \times 3 = \frac{2}{9}$ $P = \frac{2}{9} \times \frac{2}{3} = \frac{4}{27}\ (\approx 0.1481)$
Specific behaviours
✓ uses $P(\text{odd}) = \frac{2}{3}$ ✓ uses binomial expansion for 1 odd in 3 throws   ✓ calculates probability

(b) The die is thrown twice and X is the sum of the two scores.

(i) Complete the table below to show the probability distribution of X. (2 marks)

x	2	3	4	5	6
$P(X = x)$	$\mathbf{\frac{1}{4}}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{1}{9}$	$\mathbf{\frac{1}{36}}$

Solution
$P(X = 6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, $\quad P(X = 2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
Specific behaviours
✓ $P(X = 2)$ ✓ $P(X = 6)$

(ii) Determine $P(X = 3 \mid 3 \le X \le 4)$. (2 marks)

Solution
$P = \frac{1}{3} \div \left(\frac{1}{3} + \frac{5}{18}\right) = \frac{6}{11}$
Specific behaviours36
✓ $P(3 \le X \le 4)$ ✓ probability

(iii) Calculate $\mathrm{E}(X)$. (2 marks)

Solution
$\mathrm{E}(X) = \frac{9}{18} + \frac{18}{18} + \frac{20}{18} + \frac{10}{18} + \frac{3}{18} = \frac{60}{18} = \frac{10}{3}$
Specific behaviours
✓ uses sum of $x \times P(X = x)$ ✓ simplifies

Question 13 **(7 marks)**

The lifetime, T years, of a make of radio transmitter is a continuous random variable with probability density function given by

$$f(t) = 0.08e^{-0.08t}, \qquad 0 \leq t < \infty.$$

(a) Determine the probability that a randomly chosen transmitter has a lifetime of less than 20 years. (2 marks)

Solution
$P(T < 20) = \int_0^{20} f(t)\,dt$ $= 0.7981$
Specific behaviours
✓ writes integral   ✓ evaluates

(b) A company buys 18 transmitters. If they operate independently of each other, determine the probability that at least 15 of them will have lifetimes of less than 20 years. (2 marks)

Solution
$Y \sim B(12, 0.7981)$ $P(Y \geq 15) = 0.4929$
Specific behaviours
✓ indicates binomial distribution with parameters   ✓ calculates probability

(c) A transmitter has already been operating for exactly 21 years. Determine the probability that it will **not** fail within the next year. (3 marks)

Solution
$P(T > 21) = \int_{21}^{\infty} f(t)\,dt = 0.1864$ $P(T > 22) = \int_{22}^{\infty} f(t)\,dt = 0.1720$ $P(T > 22 \mid T > 21) = \dfrac{0.1720}{0.1864} = 0.9231$
Specific behaviours
✓ $P(T > 21)$ ✓ $P(T > 22)$ ✓ probability

Question 14 **(9 marks)**

(a) (i) A researcher wants to survey a proportion of individuals from the City of Melville to ask them how often they go to the cinema. They decide to go to their nearest public library and survey a group of 30 individuals. Assuming that they received responses from all 30 people, is their method of sampling appropriate? Justify your answer.

(2 marks)

Solution
Their sampling method is not appropriate as the sample is limited to people who go to the library.
Specific behaviours
✓ states that the method is not appropriate ✓ justifies their response

(ii) After receiving some complaints about the packaging of Pingu's Pizzas, a quality control manager has decided to sample every 30th box of frozen pizza on the production line to determine the proportion of pizzas which have been packaged upside down. Given that the packaging machine malfunctions at every 60th box, explain how the manager's sampling method would misrepresent the true proportion.

(2 marks)

Solution
It would be misrepresented as the proportion would be either 0 or 0.5.
Specific behaviours
✓ states that the proportion could equal to 0 ✓ states that the proportion could also equal to 0.5

A researcher wants to estimate the proportion of Western Australian teachers who are aged under 30. The researcher plans to collect sample data by visiting schools and asking teachers.

(b) Determine, to the nearest 10, the sample size the researcher should use to ensure that the margin of error of a 95% confidence interval is no more than 3%. (3 marks)

Solution
$n = \dfrac{1.96^2(0.5)(1-0.5)}{0.03^2}$ $n = 1067$ Sample size of 1070 teachers
Specific behaviours
✓ assumes $\hat{p} = 0.5$ ✓ shows sample size equation ✓ calculates n

(c) Comment on how your answer to (b) would change if the researcher had a reliable estimate that the population proportion was close to 12%. Justify your response. (2 marks)

Solution
Size of sample would decrease (*to close to 450*) as 0.12(1 – 0.12) < 0.5(1 – 0.5)
Specific behaviours
✓ states decrease ✓ justifies their response with working

Question 15 **(8 marks)**

210 black and 790 white spherical beads, identical except for their colour, are placed in a container and thoroughly mixed.

In experiment A, a bead is randomly selected, its colour noted and then replaced until a total of 16 beads have been selected.

(a) The random variable X is the number of black beads selected in experiment A. Determine $P(X > 4)$. (2 marks)

Solution
$X \sim B(16, 0.21)$ $P(X \geq 5) = 0.2327$
Specific behaviours
✓ indicates binomial RV, with parameters ✓ states P

(b) Experiment A is repeated 10 times. Determine the probability that at least one black bead is selected in each of these experiments. (2 marks)

Solution
$P(X \geq 1) = 0.9770$ $0.9770^{10} = 0.792$
Specific behaviours
✓ calculates P(at least one black) in one experiment ✓ calculates probability

In experiment B, a bead is randomly selected, its colour noted and then replaced until a total of 50 beads have been selected.

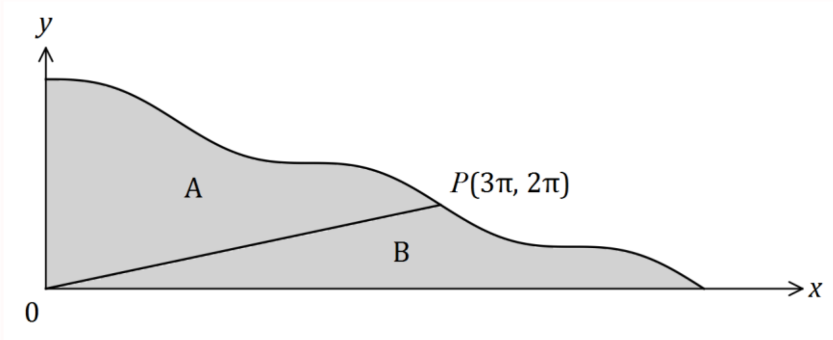
Experiments A and B are repeated a large number of times, with the proportions of black beads in each experiment, $\hat{p}_A$ and $\hat{p}_B$ respectively, recorded.

(c) Which proportion, $\hat{p}_A$ or $\hat{p}_B$, has a distribution that most likely approximates to normality? Explain your answer and state the mean and standard deviation of the normal distribution for the proportion you have chosen. (4 marks)

Solution
$\hat{p}_B$ most likely, as it is based on much larger sample size (*50 rather than 16*). Parameters: Mean: 0.21 Variance: $\frac{0.21(1-0.21)}{50} = 0.00332$, $s_x = 0.0576$
Specific behaviours
✓ chooses $\hat{p}_B$ ✓ explains $\hat{p}_B$ is based on larger sample size ✓ states mean ✓ states standard deviation

Question 16 **(7 marks)**

The curve $y = 5\pi - x + \sin x$ is shown below passing through $P(3\pi, 2\pi)$.



A straight line joins the origin to P, dividing the shaded area into two regions, A and B.

(a) Show that when $x = 5\pi$, $y = 0$. (1 mark)

Solution
$y = 5\pi - 5\pi + \sin 5\pi = 0$
Specific behaviours
✓ substitutes

(b) Determine the value of $\int_0^{3\pi} (5\pi - x + \sin x)\, dx$. (2 marks)

Solution
$I = \frac{21\pi^2}{2} + 2 \ (\approx 105.6)$
Specific behaviours
✓ evaluates integral ✓ states exact value

(c) Determine the ratio of the area of region A to the area of region B in the form $1:k$.
(4 marks)

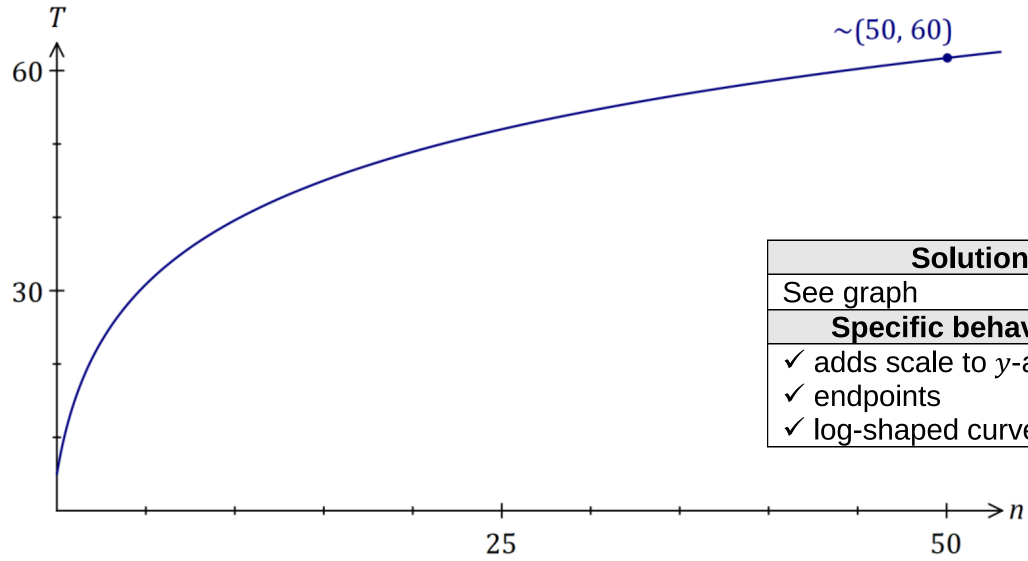
Solution
$\int_0^{3\pi} \left(\frac{2x}{3}\right) dx = 3\pi^2$ $\int_{3\pi}^{5\pi} (5\pi - x + \sin x)\, dx = 2\pi^2$ $A = \frac{21\pi^2}{2} + 2 - 3\pi^2 = \frac{15\pi^2}{2} + 2$ $B = 2\pi^2 + 3\pi^2 = 5\pi^2$ Ratio A:B is $1: 0.649$
Specific behaviours
✓ evaluates area of triangle ✓ evaluates area A ✓ evaluates area B ✓ states ratio in required form

Question 17 **(8 marks)**

Hick's law, shown below, models the average time, T seconds, for a person to make a selection when presented with n equally probable choices.

$$T = a + b\log_2(n+1), \text{ where } a \text{ and } b \text{ are positive constants.}$$

(a) Draw the graph of T vs n on the axes below when $a = 5$ and $b = 10$. (3 marks)



Solution
See graph
Specific behaviours
✓ adds scale to y-axis ✓ endpoints ✓ log-shaped curve

(b) When a pizzeria had 12 choices of pizza, the average time for patrons to make a choice was 40 seconds. After increasing the number of choices by 15, the average time to make their choice increased by 20%.

Modelling the relationship with Hick's law, predict the average time to make a choice if patrons were offered a choice of 16 pizzas. (5 marks)

Solution
$40 = a + b\log_2(12+1)$ $40 \times 1.2 = a + b\log_2(12+15+1)$ $a = 13.256, b = 7.227$ $T = 13.256 + 7.227\log_2(16+1)$ $T = 42.80 \approx 43 \text{ seconds}$
Specific behaviours
✓ writes first equation ✓ writes second equation ✓ solves for variables ✓ substitutes correctly ✓ states time, rounded to nearest second

Question 18 **(8 marks)**

The mass, X g, of wasted metal when a cast is made is a random variable with probability density function given by

$$f(x) = \begin{cases} \dfrac{2(a-x)}{a^2} & 0 \le x \le a, \\ 0 & \text{elsewhere,} \end{cases}$$

where a is a positive constant.

(a) Determine $\mathrm{E}(X)$ in terms of a. (2 marks)

Solution
$\displaystyle\int_0^a \frac{2(a-x)}{a^2} \times x\, dx = \frac{a}{3}$
Specific behaviours
✓ writes correct integral ✓ evaluates integral in terms of a

(b) The total mass of wasted metal from a random sample of 35 casts was 595 g. Estimate the value of a. (2 marks)

Solution
$\bar{x} = 595 \div 35 = 17$ $\dfrac{a}{3} = 17 \Rightarrow a = 51$
Specific behaviours
✓ calculates sample mean ✓ determines a

(c) If $a = 15$, determine

(i) $P(X \le 6)$. (1 mark)

Solution
$\displaystyle\int_0^6 \frac{2(15-x)}{15^2} dx = \frac{16}{25}$
Specific behaviours
✓ evaluates probability

(ii) $\mathrm{Var}(X)$. (3 marks)

Solution
$\mathrm{E}(X) = 15 \div 3 = 5$ $\displaystyle\int_0^{15} \frac{2(15-x)(x-5)^2}{15^2} dx = 12.5$
Specific behaviours
✓ shows value of $\mathrm{E}(X)$ ✓ writes correct integral ✓ evaluates variance

Question 19 **(7 marks)**

A polynomial function $f(x)$ is such that $\int_1^8 5f(x)\,dx = 30$.

(a) Show that $\int_8^1 f(x)\,dx = -6$. (2 marks)

Solution
$5\int_8^1 f(x)\,dx = -30$ $\int_8^1 f(x)\,dx = -6$
Specific behaviours
✓ reverses limits and changes sign ✓ factors and divides

(b) Determine the value of $\int_1^2 (2 + f(x)\,)\,dx + \int_2^8 (f(x) - 2x + 2)\,dx$. (5 marks)

Solution
$= \int_1^2 (2)\,dx + \int_1^2 \big(f(x)\big)\,dx + \int_2^8 (f(x))\,dx - \int_2^8 (2x)\,dx + \int_2^8 (2)\,dx$ $= \int_1^8 (f(x))\,dx + \int_1^8 (2)\,dx - \int_2^8 (2x)\,dx$ $= 6 + 14 - [x^2]_2^8$ $= 6 + 14 - 60$ $= -40$
Specific behaviours
✓ uses linearity to split ✓ uses interval addition ✓ integrates ✓ evaluates ✓ correct sum

Question 20 **(6 marks)**

A popcorn container of capacity 500 mL is made from paper and has the shape of an open inverted cone of radius r and height h.

Determine the least area of paper required to make the container.

Solution
$A = \pi rs = \pi r\sqrt{r^2 + h^2}$ $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$ $A = \pi r\sqrt{r^2 + \left(\frac{3(500)}{\pi r^2}\right)^2}$ $\frac{dA}{dr} = \frac{2r^6\pi^2 - 2250000}{r^2\sqrt{r^6\pi^2 + 2250000}}$ $\frac{dA}{dr} = 0 \text{ when } r = 6.963 \text{ cm}$ $A_{MIN} = 263.8 \text{ cm}^2$
Specific behaviours
✓ expresses A in terms of r and h ✓ expresses h in terms of r ✓ expresses A in terms of r ✓ differentiates A ✓ finds positive zero of derivative ✓ substitutes to find minimum area

Question 21 **(3 marks)**

A random sample of 630 wombats from a nature reserve are captured, tagged and then set free. After a suitable interval, during which time it is assumed that the wombat population does not change, another random sample of 360 wombats is caught and 27 of these are observed to be tagged.

(a) Show that a point estimate for the size of the wombat population is 8 400. (1 mark)

Solution
$\frac{630}{P} = \frac{27}{360} \Rightarrow P = 8400$
Specific behaviours
✓ shows use of direct proportion

(b) Construct a 95% confidence interval for the proportion of wombats in the population that are tagged. (2 marks)

Solution
$\hat{p} = \frac{27}{360} = 0.075, E = 0.02721, z_{0.95} = 1.96$ $(0.0478, 0.1022)$
Specific behaviours
✓ calculates margin of error ✓ states confidence interval

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

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Question number: _________

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